Solutions for Exam I, Math 10120, Fall 2016

1. Let

$$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$
$$A = \{1, 2, 3\}$$
$$B = \{2, 4, 6, 8, 10\}.$$
$$C = \{4, 5, 6, 7, 8\}.$$

Which of the following sets is equal to $(A \cap B) \cup C'$?

- {1, 2, 3, 10 }
- { 2}
- {2, 4, 5, 6, 7, 8 }
- {1, 2, 3, 9, 10 }
- { 4, 6, 8}

Solution: $A \cap B = \{2\}$, and $C' = \{1, 2, 3, 9, 10\}$, so

$$(A \cap B) \cup C' = \{2\} \cup \{1, 2, 3, 9, 10\} = \{1, 2, 3, 9, 10\}.$$

2. Consider the following sets

U = {English Words}

A = {four letter English words}

B = {English words with at least two different (see below) vowels}

C = {English words starting with the letter "m"}

Which one of the following sets contains **both** of the words "math" and "awesome" ?

- $(A \cup B) \cap C$
- $(A' \cap B) \cup C$
- $B \cap C$
- $B' \cap C' \cap A'$
- B'

Solution: The word "awesome" has more than four letters and contains at least two different vowels, so it belongs to the set $A' \cap B$. Since "math" starts with "m", it belongs to the set C. Thus the set $(A' \cap B) \cup C$ contains both words.

3. If A and B are sets in a universal set U, with $n(A) = 10$, $n(B) = 15$, $n(A \cap B') = 3$ and $n(U) = 30$, what is

$$n(A \cup B')?$$

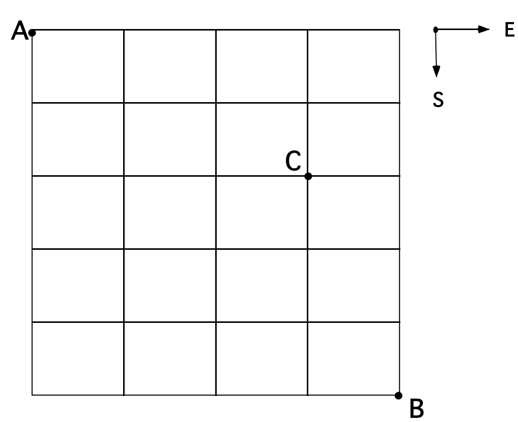
> **Solution:** When we apply the Inclusion / Exclusion Principle to the sets A and B', we get the equation
>
> $$n(A \cup B') = n(A) + n(B') - n(A \cap B').$$
>
> We know $n(A) = 10$, $n(B') = n(U) - n(B) = 30 - 15 = 15$, and $n(A \cap B') = 3$. Using this information, we solve the above equation for $n(A \cup B')$:
>
> $$n(A \cup B') = 10 + 15 - 3 = 22.$$

4. How many three-letter words, including nonsense words can be made from the letters of the word

TWEETING

assuming that letters <u>can</u> be repeated?

> **Solution:** The word TWEETING contains six distinct letters–T, W, E, I, N, and G. We use the Multiplication Principle with three steps:
>
> 1. Choose the first letter of the word–6 ways possible.
> 2. Choose the second letter. Since letters can be repeated, there are still 6 options.
> 3. Choose the third letter. Again, there are 6 possibilities.
>
> So: the Multiplication Principle tells us that there are $6 \cdot 6 \cdot 6 = 6^3$ distinct three-letter words from the letters of TWEETING, if letters may be repeated.



5. The above is a map of the roads in a small country town. A pedestrian wishes to **travel from from A to B along these roads WITHOUT going through the intersection at C**. How many routes can they choose from if they must always walk South or East?

Solution: We can identify the set of all paths from A to B where the pedestrian always walks South or East with the set $Path(A,B)$ of all rearrangements of the letters SSSSSEEEE. For instance, the rearrangement EEESSSSSE corresponds to the path in which the pedestrian starts at A, walks East for three blocks, then goes South for five blocks, and finally goes one block East to reach B. The total number of such paths is

$$n(Path(A,B)) = \frac{9!}{5!4!} = C(9,5) = 126.$$

However, we only wish to count the paths from A to B that avoid C. It is easiest to begin by counting the number of paths from A to B that cross through C–call this set $Path(A,C,B)$. A sequence in the set $Path(A,C,B)$ is one which begins with a sequence corresponding to a path from A to C, or a rearrangement of the letters SSEEE (there are $\frac{5!}{2!3!} = C(5,2) = 10$ such sequences), and ends with a sequence corresponding to a path from C to B, or a rearrangement of the letters SSSE (and there are $\frac{4!}{3!1!} = C(4,3) = 4$ of these). The Multiplication Principle tells us that

$$n(Path(A,C,B)) = 10 \cdot 4 = 40.$$

Finally, the number of paths from A to B that avoid C is

$$n(Path(A,B)) - n(Path(A,C,B)) = 126 - 40 = 86.$$

6. A squash club has 10 members; 7 women and three men. Recsports wishes to take a photograph of 5 club members consisting of three women standing in a row with two men sitting in front of them. How many such photographs are possible?

Solution:
Step 1: Choose a sequence of three women for the back row–there are $P(7,3) = 7 \cdot 6 \cdot 5 = 210$ ways to do this.
Step 2: Choose a sequence of two men to sit in the front–there are $P(3,2) = 3 \cdot 2 = 6$ such sequences.
Thus, by the Multiplication Principle, we get $P(7,3) \cdot P(3,2) = 210 \cdot 6 = 1260$ possible photographs.

7. A Chess club wants to select a team of 5 players from its ten regular players to send to a tournament. How many different teams are possible?

Solution: Since we don't care about the order in which players are selected for the team of five, we use combinations instead of permutations. The number of possible teams is then $C(10,5) = 252$.

8. A jar containing 20 jelly beans has 4 blue jelly beans, 7 red jelly beans, 3 yellow jelly beans and 6 green jelly beans. Peter selects a sample of five jelly beans from the jar. How many such samples of 5 jelly beans contain at least one yellow jelly bean?

> **Solution:** We imagine that each of the jelly beans is identified with a number from 1 to 20. A sample of 5 jelly beans is then just a subset of $\{1, \ldots, 20\}$ with five elements. The total number of such samples is $C(20, 5)$. There are 17 non-yellow jelly beans, so the number of samples of 5 jelly beans with no yellow jelly beans is $C(17, 5)$. Hence, the number of samples that contain at least one yellow jelly bean is $C(20, 5) - C(17, 5)$.

9. How many different poker hands (5 cards from a standard deck of 52) consist of two kings and 3 cards from another denomination?

   Recall there are 13 denominations in a standard deck of cards.

> **Solution:**
> **Step 1**: Pick a set of 2 kings from among the 4 kings in the deck– $C(4, 2)$ ways.
> **Step 2**: Pick a denomination other than kings– 12 possible.
> **Step 3**: Pick three cards from the four cards in the denomination you picked in Step 2– $C(4, 3)$ ways.
> Thus: there are $C(4, 2) \cdot 12 \cdot C(4, 3)$ hands consisting of 2 kings and 3 cards from another denomination.

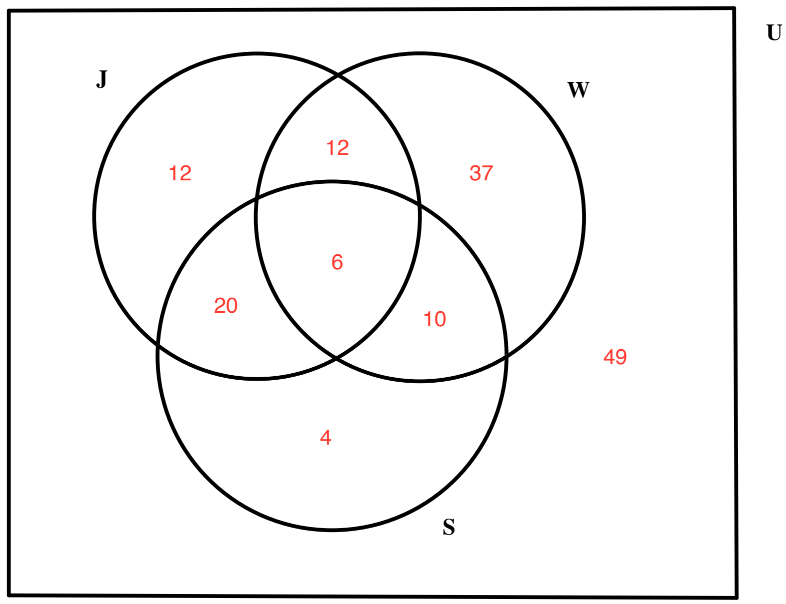
10. Bob's pizza parlor is offer a special price on a medium pizza. To order such a pizza, you must choose one type of crust from three types on offer, one type of cheese from 2 types on offer, and at most 4 toppings from a list of ten toppings on offer. (You may choose zero toppings if you wish). How many different pizzas can be ordered using the above guidelines?

> **Solution:**
> **Step 1**: Choose a crust–3 ways.
> **Step 2**: Choose the cheese–2 ways.
> **Step 3**: Choose 0, 1, 2, 3, or 4 toppings from a list of ten– there are $C(10, 0) + C(10, 1) + C(10, 2) + C(10, 3) + C(10, 4) = 1 + 10 + 45 + 120 + 210 = 386$ ways to do this.
> Thus, there are
> $$3 \cdot 2 \cdot 386 = 2316$$
> possible pizzas.

11. A group of 150 students who exercise regularly were asked whether they swam, jogged or did weight training regularly. The survey revealed that 40 swam regularly, 50 jogged regularly and 65 did weight training on a regular basis. Also 16 swam and did weight training regularly, 26 swam and jogged regularly and 18 jogged and did weight training regularly. Finally 6 of the students did all three activities regularly.

(a) Represent this information on the Venn Diagram given below.

Solution: We have $n(U) = 150$, $n(S) = 40$, $n(J) = 50$, $n(W) = 65$, $n(S \cap W) = 16$, $n(S \cap J) = 26$, $n(J \cap W) = 18$, and $n(S \cap W \cap J) = 6$. Beginning in the center and working our way out, we fill in the Venn Diagram as follows:



J = Students who jog regularly

S = Students who swim regularly

W = Students who do weight training regularly

(b) How many of the students interviewed didn't do any of the three activities regularly?

Solution: This is $n(U) - n(J \cup W \cup S) = 150 - 101 = 49$.

(c) How many of those interviewed only swam regularly?

Solution: This is $n(S \cap W' \cap J') = 4$.

12. (a) Seven runners are in a race. There are seven lanes on the track. In how many ways can the runners be assigned to the lanes, so that Jeremy is in the inside lane?

> **Solution:** Once we assign Jeremy to the inside lane, there are then $P(6,6) = 6! = 720$ ways to assign the remaining six runners to the lanes.

(b) For the runners in Part (a), prizes will we awarded to those in first, second and third place only. In how many ways can first, second and third place be awarded with Jeremy among the prizewinners.

> **Solution:**
> **Step 1**: Choose which place Jeremy gets–3 possibilities.
> **Step 2**: Fill in the other two places with other runners–$P(6,2) = 6 \cdot 5 = 30$ possibilities.
> Thus, the total number of outcomes for the awards is $3 \cdot 30 = 90$.

13. (a) A coin is flipped 6 times resulting in a sequence of heads and tails. How many of the resulting possible sequences have exactly four heads?

> **Solution:** Imagine we have six slots labelled 1 through 6. Getting a sequence with exactly four heads corresponds to picking out 4 of the slots in which to put heads. Thus there are $C(6,4) = \frac{6!}{4!2!} = 15$ possible sequences with exactly four heads.

(b) How many of the resulting possible sequences have at least four heads?

> **Solution:** A sequence has at least 4 heads if it has exactly 4 heads, exactly 5 heads, or exactly 6 heads. Thus there are $C(6,4) + C(6,5) + C(6,6) = 15 + 6 + 1 = 22$ possible sequences with at least four heads.

(c) How many of the resulting possible sequences have at most four tails?

> **Solution:** A sequence has at most four tails if it does not have five tails and it does not have six tails. The total number of sequences possible is 2^6. Thus the number of sequences with at most four tails is $2^6 - C(6,5) - C(6,6) = 57$.

14. (a) How many different words (including nonsense words) can be made by rearranging the letters of the word show below?

NUMISMATISTS

> **Solution:** There are 12 letters total in the word. The letters M, T, and I all occur twice, and the letter S occurs three times. Thus there are $\frac{12!}{2!2!2!3!}$ distinct rearrangements of the word.

(b) Peter has a collection of rare coins from various parts of the world. He has 10 United States coins, 4 Irish coins and 5 Australian coins in his rare coin collection. Peter wants to bring part of his collection to school for Show and Tell. He will bring

- at least one coin from each of the three countries,
- at most three coins from the United States,
- at most two coins from Australia and

- at least 2 Irish coins.

In how many different ways can he select the coins for Show and Tell?

Solution:
Step 1: Choose which subset of US coins to take. Peter must take at least 1 such coin and at most 3. Since he has 10 US coins total, there are $C(10,1) + C(10,2) + C(10,3) = 175$ ways to do this.
Step 2: Choose which subset of the Australian coins to take. He will take at least 1 and at most 2, and has 5 Australian coins to choose from. There are $C(5,1) + C(5,2) = 15$ ways to do this.
Step 3: Choose a subset of the Irish coins to take. This time we know he takes at least 2 of them from among a set of 4. There are $C(4,2)+C(4,3)+C(4,4) = 11$ subsets that satisfy this requirement. Finally, the Multiplication Principle tells us that the number of ways Peter can select coins from all three countries for Show and Tell is $175 \cdot 15 \cdot 11 = 28,875$.